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Intermittency trends and Lagrangian evolution of non-Gaussian statistics in turbulent flow and scalar transport

By YI LI AND CHARLES MENEVEAU
Department of Mechanical Engineering and Center of Environmental and Applied Fluid Mechanics, The Johns Hopkins University, Baltimore, MD 21218, USA

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The Lagrangian evolution of two-point velocity and scalar increments in turbulence is considered, based on the 'advected delta-vee system' (Li & Meneveau 2005). This system has already been used to show that ubiquitous trends of three-dimensional turbulence such as exponential or stretched exponential tails in the probability density functions of transverse velocity increments, as well as negatively skewed longitudinal velocity increments, emerge quite rapidly and naturally from initially Gaussian ensembles. In this paper, the approach is extended to provide simple explanations for other known intermittency trends in turbulence: (i) that transverse velocity increments tend to be more intermittent than longitudinal ones, (ii) that in two dimensions, vorticity increments are intermittent while velocity increments are not, (iii) that scalar increments typically become more intermittent than velocity increments and, finally, (iv) that velocity increments in four-dimensional turbulence are more intermittent than in three dimensions. While the origin of these important trends can thus be elucidated qualitatively, predicting quantitatively the statistically steady-state levels and dependence on scale remains an open problem that would require including the neglected effects of pressure, inter-scale interactions and viscosity.

1. Introduction

Intermittency in turbulence refers to the occurrence of rare, but intense fluctuations of small-scale quantities such as velocity increments, vorticity, strain rate, and scalar gradients. It is most typically reflected in two well-known phenomenological attributes of turbulence: (*a*) non-Gaussian tails in the probability density functions (PDF) of the small-scale quantities, and (*b*) anomalous scaling, by which scaling exponents of the moments of velocity increments deviate significantly from the prediction of Kolmogorov (1941) (see Frisch 1995). The quantitative prediction of intermittency has been a long-standing challenge in turbulence research. Much effort has been devoted to various phenomenological models for scaling of intermittency (for reviews see Frisch 1995; Sreenivasan & Antonia 1997; Yakhot & Sreenivasan 2005). Some of the models such as multifractal models (Benzi *et al.* 1984; Meneveau & Sreenivasan 1987; Frisch 1995; Chevillard, Castaing, Leveque & Arneodo 2005) and the log-Poisson model (She & Leveque 1994) have succeeded in providing good fits to many experimental results, but in general they make insufficient connection with the dynamical equations. Progress has also been made in the understanding of toy models, such as shell models (Biferale 2003) and the Kraichnan model for passive scalars (Falkovich, Gawedzki & Vergassola 2001), and with efforts to reproduce realistic features of intermittent turbulence by the superpositions of certain vortex structures (Pullin & Saffman 1998). Another related body of work has focused on the geometry of small-scale turbulence (Ashurst, Kerstein, Kerr & Gibson 1987; Lund & Rogers 1994; Tsinober, Kit & Dracos 1992). Theoretical insights into these geometric properties have been obtained from restricted Euler (RE) dynamics (Vieillefosse 1984; Cantwell 1992) through which important trends (such as preferential alignment of vorticity with the intermediate strain eigen-direction and the prevalence of axisymmetric expanding motions) can be explained from the self-stretching and rotation of the velocity gradient tensor during its Lagrangian evolution (Cantwell 1992; Zeff *et al.* 2003). Further progress has been sought by tracking the material deformation as a tool to regularize the finite-time singularities in RE dynamics (Chertkov, Pumir & Shraiman 1999; Naso & Pumir 2005), and making a direct connection between the geometry and intermittency of turbulence. Tracking of material element geometry has also been used by Jeong & Girimaji (2003), in which the Cauchy–Green tensor is used to model the viscous term.

Recently, in Li & Meneveau (2005) (referred to as LM hereafter) the evolution of the local structure of turbulence has been investigated with the goal of elucidating fundamental trends towards intermittency without having to follow all the elements of the velocity gradient tensor. Of the two phenomenological attributes of intermittency mentioned above, LM only addressed the development of the non-Gaussian tails in PDFs. The simplest analogy of the LM result is for the inviscid one-dimensional Burgers equation that describes the free motion of fluid particles (without pressure nor viscous forces). Defining $A \equiv \partial u/\partial x$, $u(x,t)$ being the velocity, one obtains $\partial A/\partial t + u\partial A/\partial x = \mathrm{d}A/\mathrm{d}t = -A^2$, which shows that initially negative velocity gradients become more negative. An initial ensemble of A with Gaussian distribution thus evolves towards a distribution having a long tail at negative A (Kraichnan 1990). Assuming that A is constant across a fixed length ℓ and defining a hypothetical velocity increment across that length as $\delta u \equiv A\ell$, the same equation can also be written as $\mathrm{d}\delta u/\mathrm{d}t = \delta\dot{u} = -\delta u^2/\ell$. In higher dimensions, the situation is more complex since several velocity components and directions are involved. In LM it was shown that the specific choice of the longitudinal and the magnitude of the transverse velocity increments across a fixed distance ℓ, along a direction advected with the flow, yields a particularly simple system of equations. Specifically, for an incompressible velocity field $\overline{u}_i(\boldsymbol{x},t)$ filtered at scale Δ and velocity gradient $\overline{A}_{ij} \equiv \partial\overline{u}_j/\partial x_i$, consider a displacement vector $\boldsymbol{r}(t)$, and the unit vector in its direction $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The longitudinal and the magnitude of the transverse components of the velocity increment vector along this direction across a fixed distance $\ell < \Delta$ are defined as

$$\delta u \equiv \ell\overline{A}_{rr} \equiv \ell\overline{A}_{ki}\hat{r}_k\hat{r}_i, \quad \delta v = \ell|P_{ij}(\boldsymbol{r})\overline{A}_{kj}\hat{r}_k|, \tag{1.1}$$

where $\overline{A}_{rr} \equiv \overline{A}_{ki}\hat{r}_k\hat{r}_i$ is the velocity gradient along the direction $\hat{\boldsymbol{r}}$, and $P_{ij}(\boldsymbol{r}) \equiv \delta_{ij} - \hat{r}_i\hat{r}_j$ is the projection operator. For a detailed sketch illustrating these definitions, see figure 1 of LM. Starting from the Navier–Stokes equations and omitting the pressure, viscous as well as subgrid-scale (SGS) forces, it was shown in LM that the Lagrangian evolution of δu and δv is described by a simple system of equations, $\delta\dot{u} = (-\delta u^2 + \delta v^2)/\ell$ and $\delta\dot{v} = -2\delta u\delta v/\ell$, called the 'advected delta-vee system' (see § 2 for details). If one defines a complex variable $z = \delta u + \mathrm{i}\delta v$, the system can be written simply as $\dot{z} = -z^2$ (as remarked in Galanti, Gibbon & Heritage (1997) in the context of a similar pair of equations to study the angle between vorticity vector and strain rate tensor). Similarly to the case of Burgers equation in one dimension, the system contains a self-amplification term for negative δu, but now the self-amplification is weakened for large transverse velocities. In the equation for δv, the cross-amplification term suggests exponential growth of δv for negative δu. In LM, DNS data were analysed to show that $\delta\dot{u}$ and $\delta\dot{v}$ predicted from this model exhibit significant correlations (correlation coefficients of 0.5–0.6) with their values measured from DNS. Furthermore, starting from initial Gaussian ensembles, the advected delta-vee system was shown to lead to the rapid emergence of exponential and stretched exponential tails in the PDFs of the velocity increments, and skewness for δu. Thus, this heavily truncated system that neglects the effects of all the forces was already sufficient to reproduce important trends of intermittency in turbulence, even though, due to the omissions of forces, no predictions could be made of the quantitative properties in the statistically steady state.

The insights into important qualitative intermittency trends provided by the advected delta-vee system motivate further investigation. As a first variant of the system, we generalize it to N dimensions and include a portion of the isotropic forces that is deterministically connected to the velocity increments. We also study the Lagrangian evolution of passive scalar increments and, in two dimensions, vorticity increments.

2. Advected delta-vee system in N-dimensions with passive scalars

The evolution of the coarse-grained velocity gradient $\overline{A}_{ij}$ (see § 1) in N dimensions is

$$\frac{\mathrm{d}\overline{A}_{ij}}{\mathrm{d}t} \equiv \frac{\partial \overline{A}_{ij}}{\partial t} + \overline{u}_m \frac{\partial \overline{A}_{ij}}{\partial x_m} = -\left(\overline{A}_{ik}\overline{A}_{kj} - \frac{1}{N}D\delta_{ij}\right) + H_{ij}, \tag{2.1}$$

in which $H_{ij} = -(\partial^2_{ij}\overline{p} - N^{-1}\delta_{ij}\partial^2_{kk}\overline{p}) - (\partial^2_{ik}\tau_{jk} - N^{-1}\delta_{ij}\partial^2_{lk}\tau_{lk}) + \nu\partial^2_{kk}\overline{A}_{ij}$ (with $\tau_{ij} \equiv \overline{u_i u_j} - \overline{u}_i\overline{u}_j$ being SGS stresses) is the anisotropic part of (the gradients of) pressure, SGS and viscous forces. The term containing $D \equiv \overline{A}_{mn}\overline{A}_{nm}$, which maintains incompressibility, is the isotropic part. Out of the $N^2 - 1$ independent components of $\overline{\boldsymbol{A}}$, the advected delta-vee system focuses on only two components associated with a direction $\hat{\boldsymbol{r}}$ that is advected by the flow. The evolution of $\boldsymbol{r}$ in a locally linear velocity field is given by

$$\dot{r}_i \equiv \mathrm{d}r_i/\mathrm{d}t = \overline{A}_{ji} r_j. \tag{2.2}$$

Taking the time derivative of δu and δv (defined by (1.1)), and using (2.1) and (2.2), one obtains the following 'advected delta-vee' system in N dimensions:

$$\delta\dot{u} = -\delta u^2\,\ell^{-1} + \delta v^2 \ell^{-1} + D\ell N^{-1} + Y, \tag{2.3}$$

$$\delta\dot{v} = -2\delta u \delta v\,\ell^{-1} + Z, \tag{2.4}$$

where $Y = \ell H_{ij}\hat{r}_i\hat{r}_j$ and $Z = \ell H_{ij}\hat{e}_j\hat{r}_i$; $\hat{\boldsymbol{e}}$ is a unit vector in the direction of the transverse velocity-increment component, perpendicular to $\hat{\boldsymbol{r}}$. The term $\ell D/N$ in (2.3) was denoted as $-2Q\ell/3$ in LM for $N = 3$, with $Q \equiv -D/2$.

For passive scalars, one begins with the coarse-grained passive scalar field $\overline{T}(\boldsymbol{x}, t)$. The evolution of its gradient vector, $\overline{G}_i \equiv \partial\overline{T}/\partial x_i$, is $\mathrm{d}\overline{G}_i/\mathrm{d}t = -\overline{A}_{ij}\overline{G}_j + K_i$, in which $K_i = -\partial^2_{ij}\Theta_j + \Gamma\partial^2_{jj}\overline{G}_i$, $\Theta_i = \overline{u_i T} - \overline{u}_i\overline{T}$ is the SGS scalar flux and Γ the scalar diffusion coefficient. Defining the scalar increment δT across ℓ along $\hat{\boldsymbol{r}}$ as $\delta T = \ell\overline{G}_k\hat{r}_k$, its Lagrangian evolution is obtained by time differentiation $\delta\dot{T} = -\delta T\delta u\ell^{-1} + W$, where $W = \ell K_i\hat{r}_i$. In two-dimensional turbulence the equation for the non-zero vorticity component ω is the same as that for a passive scalar. Thus the equation for vorticity increments, $\delta\omega = \ell\hat{r}_i\partial_i\overline{\omega}$, has the same form as for δT, but with $W = W_\omega = \ell\nu\partial^2_{kk}(\partial_i\overline{\omega})\hat{r}_i - \ell\hat{r}_i\partial^2_{ij}(\overline{u_j\omega} - \overline{u}_j\overline{\omega})$.

The equations for $\delta\dot{u}$, $\delta\dot{v}$, $\delta\dot{T}$ and $\delta\dot{\omega}$ are not closed. In LM, it was shown that (2.3) and (2.4) with $D = Y = Z = 0$ were already sufficient to display important intermittency trends. Now we take into account that of the various terms that compose D, two may be expressed as a function of $\overline{A}_{rr}$, or δu. To show this with $N = 3$, the tensor elements of $\overline{\boldsymbol{A}}$ are written in a local coordinate system defined by $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{e}}$ and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{e}}$:

$$\overline{\boldsymbol{A}} = \begin{bmatrix} \overline{A}_{rr} & \overline{A}_{re} & \overline{A}_{rn} \\ \overline{A}_{er} & \overline{A}_{ee} & \overline{A}_{en} \\ \overline{A}_{nr} & \overline{A}_{ne} & -(\overline{A}_{rr} + \overline{A}_{ee}) \end{bmatrix} \tag{2.5}$$

in which $\overline{A}_{rr} \equiv \overline{A}_{ij}\hat{r}_i\hat{r}_j = \delta u/\ell$, $\overline{A}_{re} \equiv \overline{A}_{ij}\hat{r}_i\hat{e}_j = \delta v/\ell$. Also $\overline{A}_{rn} = 0$ since the velocity increment vector has no component in the $\hat{\boldsymbol{n}}$-direction, as a consequence of the definition of $\hat{\boldsymbol{e}}$. Expressed in this frame, $D = 2\overline{A}_{rr}^2 + D^- = 2\delta u^2/\ell^2 + D^-$, where $2\overline{A}_{rr}^2$ comes from the squares of the first and the third diagonal components of $\overline{\boldsymbol{A}}$ (as written in (2.5)). D^- contains other terms in which $\overline{A}_{rr}$ and $\overline{A}_{re}$ (or δu and δv) appear as first powers, none of which is in closed form in the advected delta-vee system. Substituting D with this decomposition and neglecting X, Y, W, W_ω, and D^-, one obtains

$$\delta\dot{u} = -(1 - 2/N)\delta u^2\ell^{-1} + \delta v^2\ell^{-1}, \tag{2.6}$$
$$\delta\dot{v} = -2\delta u\delta v\ell^{-1}, \tag{2.7}$$
$$\delta\dot{T} = -\delta u\delta T\ell^{-1}, \tag{2.8}$$
$$\delta\dot{\omega} = -\delta u\delta\omega\ell^{-1}, \tag{2.9}$$

in which the last equation will be considered when $N = 2$ only. Note that another option to impose the divergence-free condition in (2.5) is to distribute $\overline{A}_{rr}$ among all remaining diagonal elements equally, i.e. with $\overline{A}_{rr}$, $\overline{A}_{ee} - \overline{A}_{rr}/2$ and $-\overline{A}_{ee} - \overline{A}_{rr}/2$ on the diagonal (for the $N = 3$ case). The prefactor in (2.6) would then change from $(N-2)/N$ to $(N-2)/(N-1)$, but resulting trends discussed in the following are not affected.

Equation (2.6) differs from the one in LM in that the first term on the right-hand side of (2.6), the self-amplification of negative δu, is weakened by the newly included portion of the pressure and SGS forces contained in D. In their discussion of conditional pressure gradient statistics, Gotoh & Nakano (2003) have also argued that the effects of pressure gradient could be modelled by a term proportional to the square of local velocity increment, to oppose the formation of Burgers-equation-like shocks. Equation (2.6) shows the cancellation is stronger for smaller N, and when $N = 2$ the self-amplification term is fully eliminated. Since the self-amplification term is responsible for generating negative skewness in the PDF of δu, this also leading to large fluctuation in δv via the cross-amplification mechanism represented by the right-hand side of (2.7) (when $\delta u < 0$), we expect both the negative skewness in the PDF of δu and the intermittency in δv to develop less rapidly at lower N, and perhaps to be absent when $N = 2$. We have also examined for $N = 3$ the correlation between $\delta\dot{u}$ modelled by (2.6) and its DNS value in the same way as in LM, finding that the correlation coefficient is almost unchanged and remains near 0.5. Equations (2.8) and (2.9) show that the intermittency in passive scalar increments and in vorticity increments (when $N = 2$) is generated by a similar cross-amplification mechanism as for the transverse velocity increment. However, as will be shown in § 3, the inherent correlations that develop between δu and δv differ from those between δu and δT (or $\delta\omega$).

3. Numerical experiments and discussions

In this section the system (2.6)–(2.9) is used to predict the evolution starting from random Gaussian initial conditions. The initial distributions of the increments δu, δT and $\delta\omega$ are taken as independent Gaussian with unit variance, while those for increment δv for $N = 2, 3, 4$ are, respectively, distributions for the magnitude of a one-, two- and three-dimensional vector with Gaussian components and unit variance. This initial ensemble could be interpreted as increments in randomly chosen directions (uniformly distributed on the N-dimensional sphere, with uniform distribution of initial solid angles, $\mathrm{d}\Omega_0$), in an N-dimensional Gaussian vector field. When the system evolves in time, the evolving directions will tend to align with expanding eigen-directions of $\overline{\boldsymbol{A}}$, and statistics over the evolving ensemble will no longer correspond to statistics taken over random directions. In order to compare model results with data that are taken at random directions not correlated with the dynamics, the model results need to be weighted with the evolving measure. Conservation of fluid volume implies that $\ell^N \mathrm{d}\Omega_0 = r(t)^N \mathrm{d}\Omega(t)$, e.g. in directions of growing $r(t)$, the solid angle $\mathrm{d}\Omega(t)$ decreases. Thus, probabilities must be weighted by $\mathrm{d}\Omega(t)/\mathrm{d}\Omega_0 = [\ell/r(t)]^N$. Solving for $r(t)$ from $\mathrm{d}r/\mathrm{d}t = r\delta u/\ell$ with $r(0) = \ell$, one obtains

$$\mathrm{d}\Omega(t)/\mathrm{d}\Omega_0 = \exp\left(-N\ell^{-1}\int_0^t \delta u(t')\,\mathrm{d}t'\right). \tag{3.1}$$

Numerically, each sample associated with an initial condition is multiplied by the factor (equation (3.1)) using $r(t)$ obtained for any particular initial condition.

Another issue concerns the status of δv as the magnitude of velocity increments, rather than a particular component as is usually reported in the literature. For example, $\delta v = (\delta v_c^2 + \delta v_d^2)^{1/2}$ when $N = 3$, where δv_c and δv_d are two orthogonal components in the plane perpendicular to $\boldsymbol{r}(t)$. As in LM, to obtain δv_c from δv, it is assumed that the transverse velocity increment vector is at a random angle with respect to a chosen transverse coordinate direction c. This assumption is justified since the displacement vector $\boldsymbol{r}$ does not have a preferred orientation with respect to a fixed frame. Then it can be shown that $\delta v_c = p\ \delta v$, where p equals 1 or -1 with equal probabilities when $N = 2$. When $N = 3$, $p = \cos\theta$ where θ is distributed uniformly in $[0, 2\pi)$. p is uniformly distributed in $[-1, 1]$ when $N = 4$. Let P_v^c and P_v be the PDFs of δv_c and δv, then

$$P_v^c(\delta v_c) = \frac{1}{2}[P_v(\delta v_c) + P_v(-\delta v_c)],\ \int_{|\delta v_c|}^{+\infty} \frac{P_v(\delta v)\,\mathrm{d}\delta v}{\pi\sqrt{\delta v^2 - \delta v_c^2}},\quad \int_{|\delta v_c|}^{+\infty} \frac{P_v(\delta v)}{2\delta v}\,\mathrm{d}\delta v \tag{3.2}$$

in two, three and four-dimensional spaces, respectively.

Without loss of generality we set $\ell = 1$. An ensemble consisting of 3×10^9 realizations is computed. The equations are integrated numerically with a fifth-order Cash–Karp Runge–Kutta scheme with adaptive stepsize control (Press, Teukolsky, Vetterling & Flannery 1992). The tolerance level is set to 10^{-4}. The PDFs are calculated by sampling within an interval $[-16, 16]$. The interval is divided into $M = 100$ bins. P_v^c is calculated using a numerical integration scheme combining the extended midpoint rule and Romberg integration (Press *et al.* 1992), with P_v approximated by linear interpolation. The upper limit of the integrals is set to the upper limit of the sampling interval, namely 16. The moments of a random variable X are calculated according to equation $\langle X^n\rangle = \sum_{i=1}^M X_i^n P_i h$, where P_i is the value of the PDF of X at X_i, h is the width of the bins.

Shown in figure 1 are the PDFs for velocity increments δu and δv_c at several times for $N = 3$, to be compared with the results in LM (for $N = \infty$). Compared with LM,

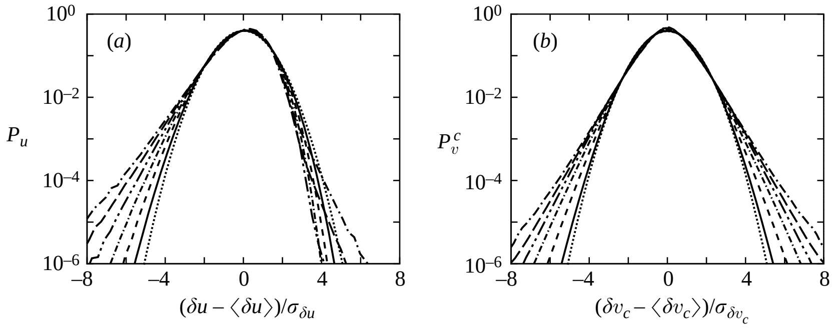

FIGURE 1. The PDFs of (*a*) δu and (*b*) δv_c in three dimensions calculated from the advected delta-vee system. Dotted lines: Gaussian, solid: $t = 0.06$, dashed: 0.12, dash-dotted: 0.18, dash-double-dotted: 0.24, long dashed: 0.30 and long dash-dotted: 0.36.

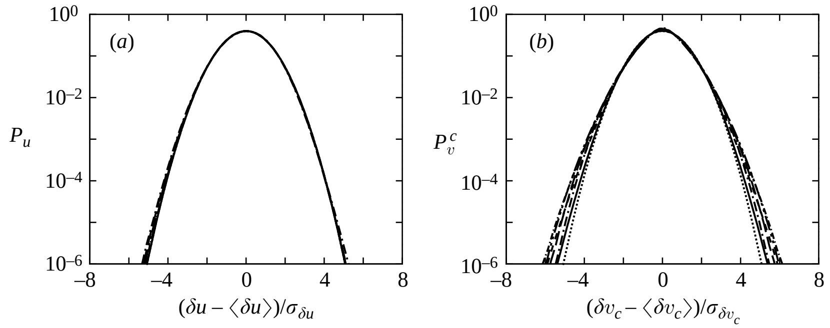

FIGURE 2. As figure 1 but for $N = 2$.

the evolution of the PDFs is slower due to the cancellation from the retained deterministic terms in D, but the qualitative trends remain the same. There still is rapid development of negative skewness of the PDFs for δu, and the exponential tails in the PDFs for δv_c. Remarkably, for $N = 2$, the PDFs for δu calculated from the model are nearly identical to Gaussian (figure 2*a*). The PDFs for δv_c, shown in figure 2(*b*), develop tails that are slightly wider than Gaussian, but at no time do they stretch sufficiently to approach exponential. The absence of non-Gaussian tails in two dimensional turbulence is in general agreement with DNS and experimental observations (Paret & Tabeling 1998; Boffeta, Celani & Vergassola 2000). The PDFs for the velocity increments for $N = 4$ are shown in figure 3. Compared to those for $N = 3$, the tails are wider at any given instant, consistent with the DNS results reported recently by Suzuki *et al.* (2005), in which they found the intermittency of velocity increments is stronger in four dimensional turbulence than in three dimensional one. PDFs of δv_c for different N are compared in figure 4(*a*) at $t = 0.24$, which clearly shows that the tails are wider at higher N. We also compare in figure 4 the skewness of δu, $S(\delta u) \equiv \langle (\delta u - \langle \delta u \rangle)^3 \rangle / \langle (\delta u - \langle \delta u \rangle)^2 \rangle^{3/2}$, and the flatness $F(X) \equiv \langle (X - \langle X \rangle)^4 \rangle / \langle (X - \langle X \rangle)^2 \rangle^2$ for $X = \delta u$ and δv_c. For $N = 2$, the skewness and flatness factor of δu remain at their initial Gaussian values, clearly showing that no intermittency develops in δu. As for δv_c, the PDFs show no exponential tails, whereas the flatness does increase above the Gaussian value $F = 3$. As is evident in figure 2(*b*), the deviation from the Gaussian value is not caused by exponential tails but by slight deformation of the PDFs. For $N = 3$, we observe that $F(\delta u)$ remains significantly smaller than $F(\delta v_c)$, consistent with the trend for unfiltered velocity gradients in DNS

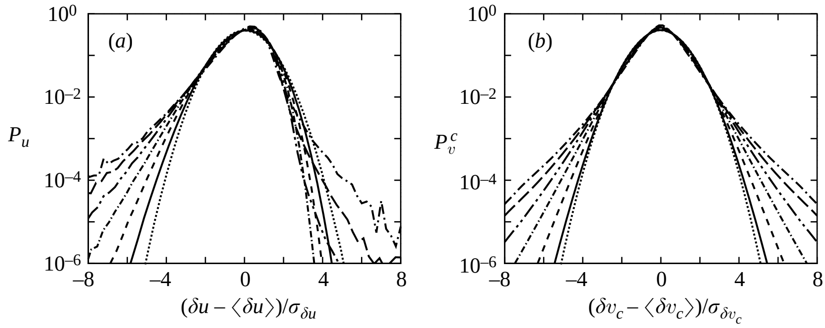

FIGURE 3. As figure 1 but for $N = 4$.

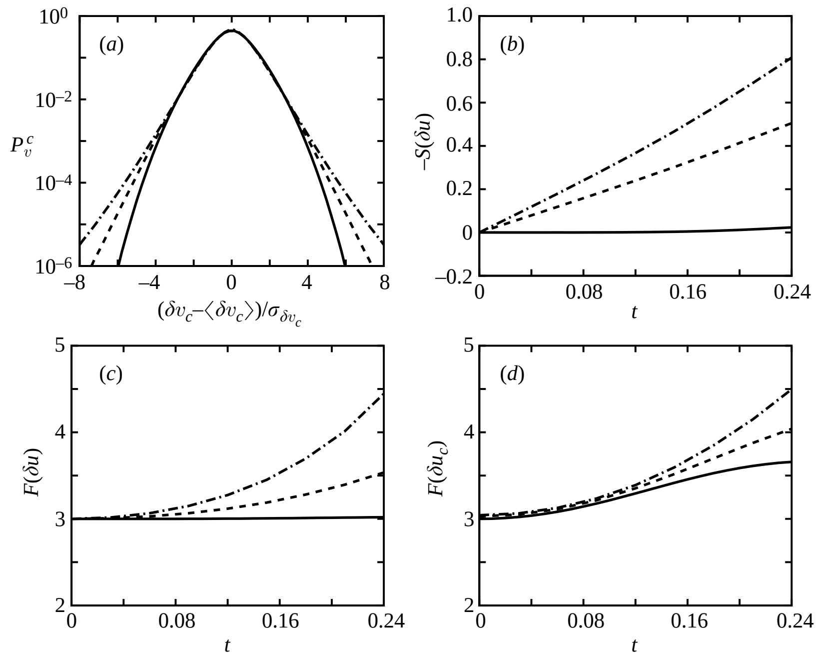

FIGURE 4. (*a*) Comparison of the PDFs of δv_c at $t = 0.24$ in two (solid line), three (dashed) and four dimensions (dash-dotted). (*b–d*) Time evolution of the skewness for δu, the flatness factors of δu and δv_c respectively. Legend same as (*a*).

(Gotoh, Fukayama & Nakano 2002) that consistently shows stronger intermittency of transverse velocity increments compared to longitudinal ones. The same trend is also observed for $N = 4$. The lower intermittency in δu compared to δv_c is due to the $(-2/N)$ term that causes cancellation of self-stretching due to incompressibility, whereas no such term occurs in the transverse direction. Figure 4(*b*) shows that the skewness is also higher for $N = 4$ than for $N = 3$, consistent with the results of Suzuki *et al.* (2005). As mentioned before, the system cannot tend to a steady state due to the omission of several force terms. Thus the skewness grows significantly below −0.5 (or $-0.3 \sim -0.4$ in the inertial range, see Cerutti, Meneveau & Knio (2000)). For $N = 3$, when the skewness crosses through −0.5 at about $t = 0.24$, we note from figure 4 that $F(\delta u) \approx 3.5$ and $F(\delta v_c) \approx 4.0$.

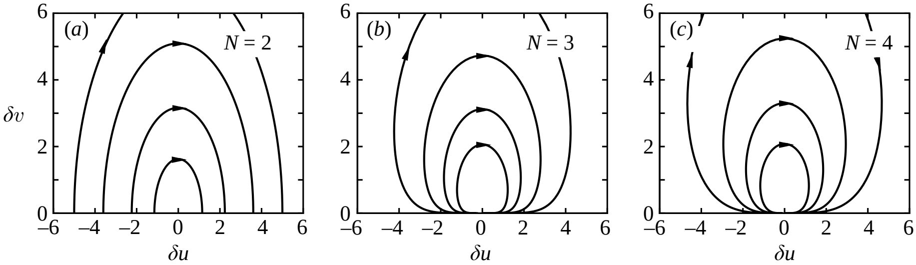

FIGURE 5. Trajectories in $(\delta u, \delta v)$ phase space for (*a*) $N=2$, (*b*) $N=3$ and (*c*) $N=4$.

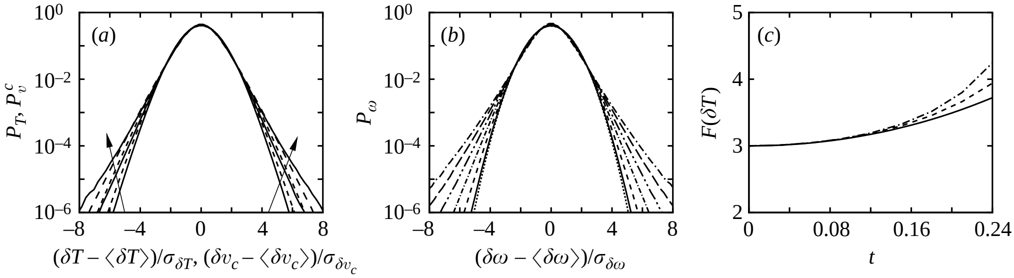

FIGURE 6. (*a*) Comparison of the PDFs of δT with δv_c in three dimensions. Solid lines: δT, dashed: δv_c. Along the arrow: $t=0.12$, 0.18 and 0.24. (*b*) Evolution of PDFs of $\delta\omega$ in two dimensions. Legend same as figure 1. (*c*) Evolution of the flatness factor of δT for $N=2$ (solid line), 3 (dashed) and 4 (dash-dotted).

The basic trends in the PDFs can be understood from the invariant of (2.6) and (2.7), which is found to be $U=[\delta u^2+N(N+2)^{-1}\delta v^2]\delta v^{2/N-1}$. In LM (where $N=\infty$) it was argued that the δv^{-1} factor is responsible for the subsequent rapid growth of δv when it is small initially. The same is true qualitatively for $N>2$, but evidently not so for $N=2$ when the factor disappears. Some trajectories in the $(\delta u, \delta v)$ phase space for different U are drawn in figure 5. Obviously, the faster expansion of the trajectories for $N=4$ when evolving towards negative δu is associated with higher intermittency in the tails of the PDFs of δv_c. When $N=2$ the trajectories fall on ellipses, along which δu can only increase. Therefore no wider tails in the PDFs of δu are observed.

Next, the evolution of PDFs for passive scalar increments and, when $N=2$, vorticity increments is considered. For $N=3$, PDFs of δT are compared to those of δv_c in figure 6(a) at three times, focusing on the tails of the PDFs. A crossover behaviour is observed: the tails of PDFs of δT develop slower initially, but overtake those of δv_c at about $t=0.18$ and remain more stretched later on. Thus it appears that the system predicts more intense intermittency for scalars than velocities, once correlations develop between δu and δv that inhibit the growth in δv (because high δv causes negative δu to grow less rapidly, whereas no such feedback from δT on δu occurs). Higher scalar intermittency compared to that of velocity is consistent with experimental results of, e.g., Antonia, Hopfinger, Gagne & Anselmet (1984). For $N=2$, we consider the evolution of the PDFs of $\delta\omega$. Figure 6(*b*) shows that the tails indeed gradually build up as time progresses, attaining an exponential and then slightly stretched exponential shape, in qualitative agreement with the experimental results in Paret, Jullien & Tabeling (1999). Thus even when no skewness exists for δu, negative values of δu yield rapid growth of the tails of $\delta\omega$, while for δv the same does not occur due to the feedback onto δu. Finally, shown in figure 6(*c*) is the evolution of the flatness factor of δT for different N, which shows that intermittency of δT increases with N.

4. Conclusions

The advected delta-vee system obtained in Li & Meneveau (2005) has been generalized to arbitrary dimensions, including taking account of parts of the forces that originate from the incompressibility condition. These parts provide partial cancellation of the self-amplification of negative longitudinal velocity increments. Implications on the development, out of Gaussian initial conditions, of intermittency in the PDFs of velocity increments, passive scalar increments and, in two dimensions, vorticity increments, have been explored via numerical experimentation. A main finding is that the intermittency (non-Gaussian, flared-up tails) in the PDFs is stronger in higher dimensions. In two dimensions the cancellation of self-amplification is complete, eliminating the generation of non-Gaussian tails. The cause for the intermittency of passive scalar increments and, in two dimensions, vorticity increments, is shown to be the same cross-amplification mechanism by which intermittency in transverse velocity increments is generated, but without the restituting term that slows the growth of the latter. In two dimensions the PDFs of vorticity increments develop tails that stretch out wider than Gaussian, and in three dimensions the flatness of transverse velocity increments predicted from the system is higher than that of longitudinal ones. Also, the tails of the PDFs of passive scalar increments in three dimensions reach a higher level than those of transverse velocity increments, after an initial period of evolution. All these trends are consistent with prior DNS and experimental results. The fact that the model with SGS effects neglected qualitatively reproduces many observed intermittency trends suggests that at any given scale ℓ, the trends towards intermittency only require self-interactions at the same scale and larger, while interactions with small-scale SGS terms (neglected here) are needed to help regularize the statistics so that the dynamics become stationary at that scale.

In summary, the results consolidate the view that the advected delta-vee system may be used to provide relatively simple dynamical explanations for a number of intermittency trends observed in turbulence. Together with the success of restricted Euler dynamics in explaining geometric alignment trends, these results show that considering the evolution in a Lagrangian sense may continue to provide valuable new insights into natural trends of turbulence. However, as a consequence of neglecting the contributions from D^- and the anisotropic pressure Hessian, inter-scale interactions and viscous damping, the system does not reach statistically steady state, and thus no predictions of dependence on length-scale and associated scaling exponents can be made. More knowledge about the neglected terms is required in order to enable such more quantitative predictions.

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REFERENCES

Antonia, R. A., Hopfinger, E. J., Gagne, Y. & Anselmet, F. 1984 Temperature structure functions in turbulent shear flows. *Phys. Rev.* A **30**, 2704–2707.

Ashurst, W. T., Kerstein, A. R., Kerr, R. M. & Gibson, C. H. 1987 Alignment of vorticity and scalar gradient with strain rate in simulated Navier–Stokes turbulence. *Phys. Fluids* **30**, 2343–2353.

Benzi, R., Paladin, G., Parisi, G. & Vulpiani, A. 1984 On the multifractal nature of fully developed turbulence and chaotic systems. *J. Phys.* A **17**, 3521–3531.

Biferale, L. 2003 Shell models of energy cascade in turbulence. *Annu. Rev. Fluid Mech.* **35**, 441–468.

Boffeta, G., Celani, A. & Vergassola, M. 2000 Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behavior. *Phys. Rev.* E **61**, R29–R32.

Cantwell, B. J. 1992 Exact solution of an restricted Euler equation for the velocity gradient tensor. *Phys. Fluids* A **4**, 782–793.

Cerutti, S., Meneveau, C. & Knio, O. M. 2000 Spectral and hyper eddy viscosity in high-Reynolds-number turbulence. *J. Fluid Mech.* **421**, 307–338.

Chertkov, M., Pumir, A. & Shraiman, B. I. 1999 Lagrangian tetrad dynamics and the phenomenology of turbulence. *Phys. Fluids* **11**, 2394–2410.

Chevillard, L., Castaing, B., Leveque, E. & Arneodo, A. 2005 Unified multifractal description of velocity increments statistics in turbulence. *Physica D* (in press).

Falkovich, G., Gawedzki, K. & Vergassola, M. 2001 Particles and fields in fluid turbulence. *Rev. Mod. Phys.* **73**, 913–975.

Frisch, U. 1995 *Turbulence: the Legacy of A. N. Kolmogorov*. Cambridge University Press.

Galanti, B., Gibbon, J. D. & Heritage, M. 1997 Vorticity alignment results for the three-dimensional Euler and Navier-Stokes equations. *Nonlinearity* **10**, 1675–1694.

Gotoh, T., Fukayama, D. & Nakano, T. 2002 Velocity statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation. *Phys. Fluids* **14**, 1065–1081.

Gotoh, T. & Nakano, T. 2003 Role of pressure in turbulence. *J. Statist. Phys.* **113**, 855–874.

Jeong, E. & Girimaji, S. S. 2003 Velocity-gradient dynamics in turbulence: Effect of viscosity and forcing. *Theoret. Comput. Fluid Dyn.* **16**, 421–432.

Kolmogorov, A. N. 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Dokl. Akad. Nauk. SSSR* **30**, 301–305.

Kraichnan, R. H. 1990 Model of intermittency in hydrodynamic turbulence. *Phys. Rev. Lett.* **65**, 575–578.

Li, Y. & Meneveau, C. 2005 Origin of non-Gaussian statistics in hydrodynamic turbulence. *Phys. Rev. Lett.* **95**, 164502 (referred to herein as LM).

Lund, T. S. & Rogers, M. M. 1994 An improved measure of strain state probability in turbulent flows. *Phys. Fluids* **6**, 1838–1847.

Meneveau, C. & Sreenivasan, K. R. 1987 Simple multifractal cascade model for fully developed turbulence. *Phys. Rev. Lett.* **59**, 1424–1427.

Naso, A. & Pumir, A. 2005 Scale dependence of the coarse-grained velocity derivative tensor structure in turbulence. *Phys. Rev.* E **72**, 056318-1-056318-9.

Paret, J., Jullien, M.-C. & Tabeling, P. 1999 Vorticity statistics in the two-dimensional enstrophy cascade. *Phys. Rev. Lett.* **83**, 3418–3421.

Paret, J. & Tabeling, P. 1998 Intermittency in the two-dimensional inverse cascade of energy: experimental observations. *Phys. Fluids* **10**, 3126–3136.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. & Flannery, B. P. 1992 *Numerical Recipes in Fortran 77*. Cambridge University Press.

Pullin, D. I. & Saffman, P. G. 1998 Vortex dynamics in turbulence. *Annu. Rev. Fluid Mech.* **30**, 31–51.

She, Z.-S. & Leveque, E. 1994 Universal scaling laws in fully developed turbulence. *Phys. Rev. Lett.* **72**, 336–339.

Sreenivasan, K. R. & Antonia, R. A. 1997 The phenomenology of small-scale turbulence. *Annu. Rev. Fluid Mech.* **29**, 435–472.

Suzuki, E., Nakano, T., Takahashi, N. & Gotoh, T. 2005 Energy transfer and intermittency in four-dimensional turbulence. *Phys. Fluids* **17**, 081702.

Tsinober, A., Kit, E. & Dracos, T. 1992 Experimental investigation of the field of velocity-gradients in turbulent flows. *J. Fluid Mech.* **242**, 169–192.

Vieillefosse, P. 1984 Internal motion of a small element of fluid in an inviscid flow. *Physica* A **125**, 150–162.

Yakhot, V. & Sreenivasan, K. R. 2005 Anomalous scaling of structure functions and dynamic constraints on turbulence simulations. *J. Stat. Phys.* **121**, 823–841.

Zeff, B. W., Lanterman, D. D., McAllister, R. *et al.* 2003 Measuring intense rotation and dissipation in turbulent flows. *Nature* **421**, 146–149.